MATH2230A

Tutorial 11 22-11-2016

Topic: Cauchy's Residue theorem

Thm: (Cauchy's Residue theorem)

- Let C be positively oriented simple closed contour.
  If f is analytic on and inside C except finitely many singular points $z_1, z_2, \cdots, z_k$, then we have

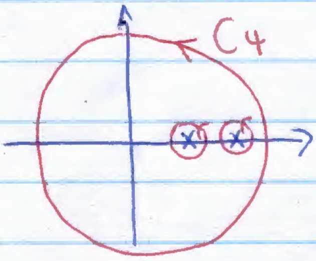
$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z)$$

Residue can be found by using (I) Cauchy Integral formula or
(II) Laurent series expansion.

Example: 1) Consider the integral

$$\int_{C_4} \frac{1}{z^2-5z+6}\,dz \quad , \text{where } C_4 = \{z \mid |z| = 4\}$$

Ans:



$$\int_{C_4} \frac{dz}{z^2-5z+6} = \int_{C_4} \frac{dz}{(z-2)(z-3)}$$

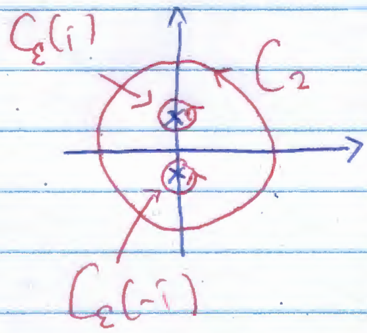
$$= -\int_{C_4} \frac{dz}{z-2} + \int_{C_4} \frac{dz}{z-3}$$

$$= -2\pi i + 2\pi i$$

$$= 0.$$

2) Find the integral

$$\int_{C_2} \frac{e^{az}}{1+z^2}\,dz \quad , \text{where } a \in \mathbb{R}.$$



Ans:

$$\int_{C_2} \frac{e^{az}}{1+z^2}\,dz = \int_{C_2} \frac{e^{az}}{(z+i)(z-i)}\,dz$$

$$= \int_{C_\varepsilon(i)} \frac{e^{az}/(z+i)}{z-i}\,dz + \int_{C_\varepsilon(-i)} \frac{e^{az}/(z-i)}{z+i}\,dz$$

By Cauchy's Integral Formula $\rightarrow$

$$= \left(\frac{e^{ai}}{2i} \mp \frac{e^{-ai}}{-2i}\right)(2\pi i)$$

$$= (2\pi i)\left(\frac{e^{ai} - e^{-ai}}{2i}\right)$$

$$= 2\pi i \sin a$$

3) Find the residue of the function

$$f(z) = \frac{1}{z}\sin^2\frac{1}{2z} \quad \text{at } z = 0$$

<u>Ans</u>: Recall that last time we show that

$$f(z) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!\, z^{2n+1}}$$

Hence $\quad \mathrm{Res}_{z=0}(f(z)) = \text{coefficient of } z^{-1}$

$$= 0$$

4) Find $\displaystyle\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{a^n}$, where $a > 4$, by assuming that the summation sign commute with the contour integral.

<u>Ans</u>: Recall that $\quad (1+z)^n = \displaystyle\sum_{\ell=0}^{n}\binom{n}{\ell} z^\ell$

By Cauchy Integral formula, we have

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{(1+z)^n}{z^{\ell+1}}\,dz = \binom{n}{\ell}$$

Hence

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{a^n} = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{|z|=1}\frac{(1+z)^{2n}}{a^n z^{n+1}}\,dz$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{|z|=1}\left(\frac{(1+z)^2}{az}\right)^n\frac{dz}{z}$$

$$= \frac{1}{2\pi i}\int_{|z|=1}\sum_{n=0}^{\infty}\left(\frac{(1+z)^2}{az}\right)^n\frac{dz}{z}$$

$$= \frac{1}{2\pi i} \int_{|z|=1} \frac{1}{1 - \frac{(1+z)^2}{az}} \frac{dz}{z}$$

$$= \frac{1}{2\pi i} \int_{|z|=1} \frac{a\,dz}{az - (1+z)^2}$$

$$= \frac{a}{2\pi i} \int_{|z|=1} \frac{dz}{(-z^2 + (a-2)z - 1)}$$

Note that $-z^2 + (a-2)z - 1 = 0$

$$\Leftrightarrow \quad z = \frac{-(a-2) \pm \sqrt{(a-2)^2 - 4(-1)(-1)}}{(-2)}$$

$$\Leftrightarrow \quad = \frac{(a-2) \pm \sqrt{a^2 - 4a}}{2}$$

Denote the two solutions by z_+ and z_-

Then we have

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{a^n} = \frac{a}{2\pi i} \int_{|z|=1} \frac{dz}{(-z^2 + (a-2)z - 1)}$$

$$= \frac{-a}{2\pi i} \int_{|z|=1} \frac{dz}{(z - z_+)(z - z_-)}$$

Note that

$$|z_+| = \left| \frac{(a-2) + \sqrt{a^2 - 4a}}{2} \right|$$

$$> \left| \frac{a-2}{2} \right|$$

$$> \left| \frac{4-2}{2} \right|$$

$$= 1$$

$$|z_-| = \left| \frac{(a-2) - \sqrt{a^2 - 4a}}{2} \right|$$

$$= \left| \frac{(a-2)^2 - (a^2 - 4a)}{2((a-2) + \sqrt{a^2 - 4a})} \right|$$

$$= \left| \frac{2}{(a-2)+\sqrt{a^2-4a}} \right|$$

$$< \frac{2}{a-2}$$

$$< 1$$

Hence by Cauchy's Residue thm,

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{a^n} = \frac{-a}{2\pi i} \int_{|z|=1} \frac{dz/(z-z_+)}{z-z_-}$$

$$= \frac{-a}{2\pi i} (2\pi i) \operatorname*{Res}_{z=z_-} \left(\frac{1}{z-z_+} \right)$$

$$= \frac{a}{z_+ - z_-}$$

$$= \frac{a}{\sqrt{a^2-4a}} \text{ //}$$

5) Let P be a polynomial of degree ≥ 2.
Let $z_1, z_2, \ldots, z_k$ be distinct roots of P.
Let $\frac{1}{p(z)} = \frac{A_1}{z-z_1} + \cdots + \frac{A_k}{z-z_k} +$ terms of the form $\frac{B}{(z-z_l)^j}$
where $j \geq 2$.

a) Show that $A_1 + A_2 + \cdots + A_k = 0$. (Hint: Consider $\lim_{R\to\infty} \int_{|z|=R} \frac{dz}{p(z)}$)

b) Hence compute the integral

$$\int_{|z|=2} \frac{dz}{(z^{2230}+1)(z-3)}$$

Ans: a) First, suppose we have $\lim_{R\to\infty} \int_{|z|=R} \frac{dz}{p(z)} = 0$ —— (*)

Since $\forall R > \max\{|z_1|, |z_2|, \ldots, |z_k|\}$,

$$\int_{|z|=R} \frac{dz}{p(z)} = 2\pi i \sum_{i=1}^{k} A_i$$

We have

$$\sum_{i=1}^{k} A_i = \frac{1}{2\pi i} \int_{|z|=R} \frac{dz}{p(z)}$$

$$= \lim_{R\to\infty} \frac{1}{2\pi i} \int_{|z|=R} \frac{dz}{p(z)}$$

$$= 0.$$

$a_n \neq 0$

It remains to show $(*)$. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

Since $\displaystyle \lim_{z\to\infty} \left| \frac{a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}{a_n z^n} \right| = 0$,

$\exists R_0 > 0$ s.t. $\displaystyle \left| \frac{a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}{a_n z^n} \right| < \frac{1}{2} \quad \forall |z| > R_0.$

$$\Rightarrow |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| < \frac{|a_n|}{2} |z|^n.$$

So

$$\left| \frac{1}{p(z)} \right| \leq \frac{1}{|a_n z^n| - |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0|}$$

$$\leq \frac{2}{|a_n||z|^n}$$

$$= \frac{2}{|a_n| R^n} \quad \text{for } |z| > R_0$$

and

$$\left| \int_{|z|=R} \frac{1}{p(z)} dz \right| \leq (2\pi R)\left(\frac{2}{|a_n| R^n} \right)$$

$$= \frac{4\pi}{|a_n| R^{n-1}}$$

$$\to 0 \quad \text{as } R \to \infty,$$

since $n \geq 2$ by assumption.

This shows $(*)$.

b) Let $z_1, z_2, \cdots, z_{2230}$ be all the distinct roots of the equation $z^{2230}+1=0$.
Let $z_{2231}=3$.
By a), we have $\sum_{i=1}^{2230} A_i = -A_{2231}$

By Residue thm, we have

$$\int_{|z|=2} \frac{dz}{(z^{2230}+1)(z-3)} = 2\pi i \sum_{i=1}^{2230} \operatorname{Res}_{z=z_i} \left(\frac{1}{(z^{2230}+1)(z-3)}\right)$$
$$= 2\pi i \sum_{i=1}^{2230} A_i$$
$$= 2\pi i \left(-A_{2231}\right)$$
$$= -\int_{|z-3|=\varepsilon} \frac{1/(z^{2230}+1)}{z-3}\, dz$$
$$= \frac{-2\pi i}{3^{2230}+1}$$ //

By Cauchy's Integral formula.